MHD Equilibrium Reconstruction from Correlated Experimental Data

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Reconstruction of magnetohydrodynamics (MHD) equilibria in tokamaks and other magnetic fusion confinement devices is an important tool in diagnosing the plasma behavior. Axisymmetric MHD equilibria are solutions of the Grad-Shafranov (GS) equation (see Fig. 1). Reconstruction proceeds as a nonlinear least squares problem, i.e., minimizing

$$\chi^2 = \frac{1}{2}\sum_{ij} \left(B_i^m - B_i^t\left(\vec{\alpha},\vec{\beta}\right)\right) D_{ij}\; B_j^m\left(-B_j^t\left(\vec{\alpha},\vec{\beta}\right)\right)$$

with respect to GS parameters α_n, β_n, where B_i^m and $B_i^t(\vec{\alpha},\vec{\beta})$ are the measured and theoretical poloidal magnetic field, as in Fig. 1. Theoretical values come from the GS equation with α_n parameterizing the toroidal field and β_n parameterizing the pressure. Also, D_{ij} is the inverse of the covariance matrix of the measured data, assumed to be gaussian. Off-diagonal terms in the matrix D are related to data correlations, and our investigations show that in tokamaks these correlations can seriously influence the reconstruction, compared to the commonly used approach which keeps only the diagonal terms. This is particularly true when the correlations have a power-law behavior in distance.

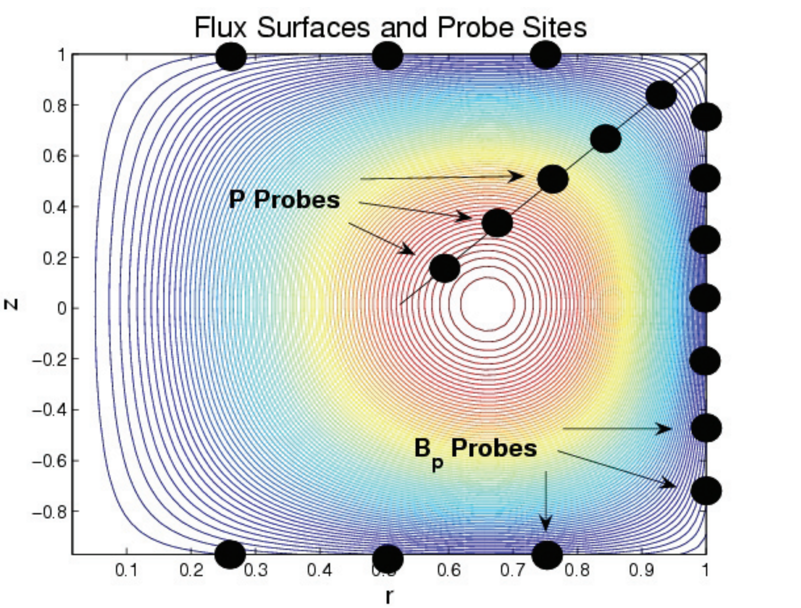

***Figure 1—** Flux surfaces for a solution of the GS equation, showing edge poloidal magnetic field measurements as well as internal pressure measurements.*

The Bayesian interpretation is that the posterior distribution of α_n, β_n is related to the likelihood of the data

$f_1(\vec{\alpha},\vec{\beta}\,|\,B_i^m) \propto f_2(B_i^m\,|\,\vec{\alpha},\vec{\beta})$. A uniform prior is assumed. The maximum of $f_1(\vec{\alpha},\vec{\beta}\,|\,B_i^m)$ is found by maximizing the likelihood $f_2(B_i^m\,|\,\vec{\alpha},\vec{\beta}) \propto e^{-\chi^2}$, or minimizing $\chi^2(\vec{\alpha},\vec{\beta})$.

Because of the nonlinear nature of the field measurements $B_j^t(\vec{\alpha},\vec{\beta})$ from GS, we must minimize χ^2 by numerical means such as simulated annealing rather than by simpler approaches such as singular value decomposition (SVD). For low noise, the posterior $f_1(\vec{\alpha},\vec{\beta}\,|\,B_i^m)$ is gaussian in α_n, β_n and is characterized by the posterior covariance matrix. (In this limit, the estimate is obtained nonlinearly but the posterior covariance matrix is found linearly, the extended Kalman filter.) For higher noise levels, however, the posterior is not gaussian and its covariance matrix must be determined numerically.

The reconstruction has a number of near-degeneracies if only the edge poloidal magnetic field is measured. Terms of the form $(\alpha_n + r^2\beta_n)$ occur in GS and for large *n* are approximated by $(\alpha_n + r_p^2\beta_n)$, where r_P is the radius of the plasma center (Fig. 1), because for large *n* the current density is peaked near the center. This is a near degeneracy between pressure and toroidal field. Since the typical tokamak current density profile is peaked, this near-degeneracy can be important. This effect is quantified in Fig. 2, in which we have solved GS with $n = 1, 2, 3, 4$. We have then plotted the χ^2 contours [contours of the posterior distribution function $f_1(\alpha_n,\beta_n\,|\,B_i^m) \propto e^{-\chi^2}$] for each *n*. It is clear that as *n* increases, the stiffness increases, with f_1 aligning along $(\alpha_n + r_p^2\beta_n) = const$. In the linear regime, this stiffness is expressed as the ratio of the eigenvalues of the posterior covariance matrix. For weak nonlinearity, the inverse of the Hessian of lnf_1 can be used.

It is possible to break this degeneracy by making direct pressure measurements in the plasma interior (by Thomson scattering). This is indicated in Fig. 1. Since the pressure *gradient* occurs in GS, pressure differences are required. Including these pressure differences into χ^2 leads to results shown in Fig. 3. We plot the $p = 1/2$ curve, i.e., the contour of $f_1(\alpha_n, \beta_n)$ within which half the probability lies. For a single pair of pressure measurements this curve collapses in the β_n (pressure) direction. For six pressure measurements the curve narrows slightly more. Interestingly, the accuracy in the α_n parameter is increased by the first pair of pressure measurements, but hardly at all by the subsequent measurements.

Modifications based on these quantitative considerations are being incorporated into equilibrium reconstruction codes and experimental studies are being initiated to determine correlations of external magnetic measurements.

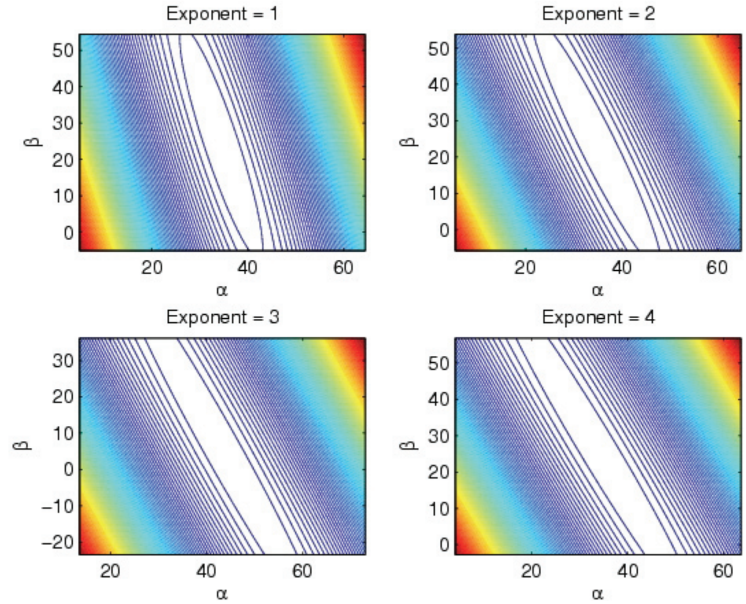

Figure 2—
Contours of χ^2, i.e., contours of the posterior distribution function, with exponents n = 1, 2, 3, 4.

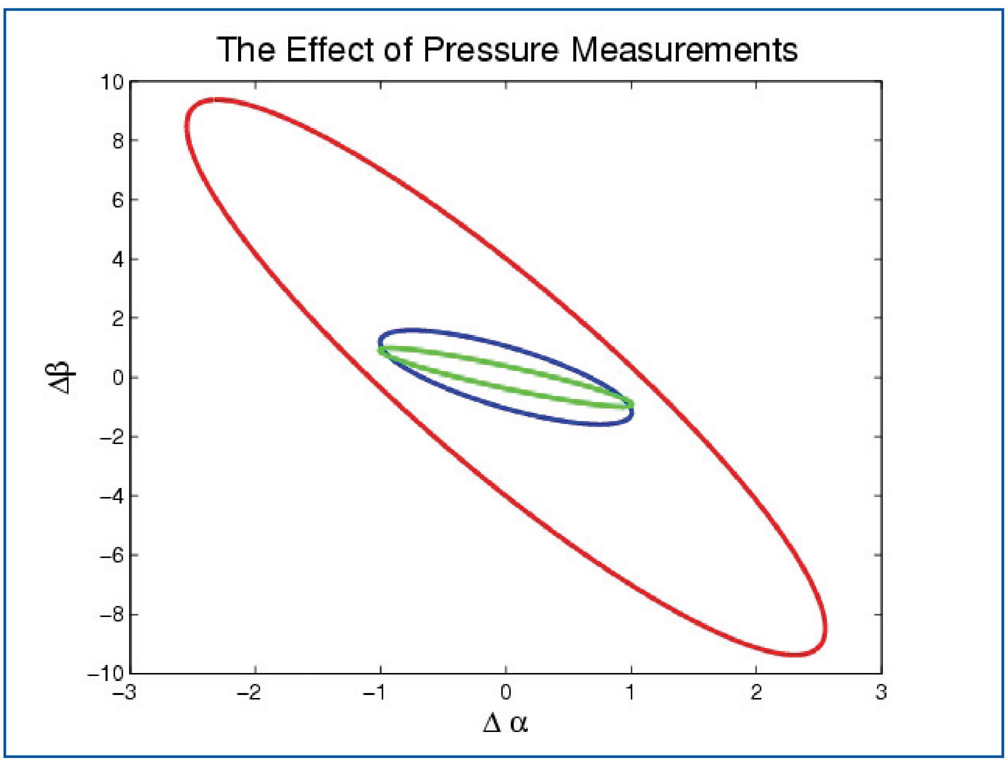

Figure 3—
Probability p = 1/2 curves in $(\Delta\alpha_n, \Delta\beta_n)$ with no internal pressure measurement (red), with one pressure difference (blue), and with five pressure differences (green).